

An Approximate Method for Treating a Class of Multiqueue Problems

Abstract: The following problem is considered: N queues of unrestricted length are served in cyclic order by a single server. Input to each queue is Poisson, the service time distribution may be arbitrary, and a finite time is required by the server to go from one queue to the next. Supposing that at any queue the server serves all units which he finds when he arrives, what is the probability p_n that in a stationary state he finds exactly n units? The method for solving this problem is based on the notion of a "self-consistent" probability distribution and is actually applicable to a general class of multiqueue situations of which the one considered here is typical.

1. Introduction

Recently, papers devoted to queueing problems have appeared with increasing frequency.¹ By now, problems involving only a single queue have received an almost exhaustive discussion and some attention has turned to the analysis of systems containing many queues.

The difficulty in treating such systems is this: A complete description of their state means the specification of joint probabilities depending upon a large number of indices, each index in general corresponding to a queue. Hence, the determination of these probabilities usually demands the solution of a large number of sets of equations. However, rarely is such a complete description required. Generally, one is interested only in the probability distribution of the length of any single queue, or the probability distribution of the waiting time or some other distribution or parameter which gives a more or less simple characterization of the system. But as a rule, it is impossible to write down an equation containing only these quantities and satisfied *exactly* by them, i.e., one cannot obtain an exact partial description of the system without first obtaining an exact complete description.

Here we present an approximate method for treating a class of problems, involving many queues, which has arisen in connection with the design of a multiterminal communication system. However, we believe that this method is more widely applicable. Phrased in the traditional language of queueing theory, the situation in which we are interested is the following:

We have N queues with a single server. The input to any single queue is independent of the input to any other queue and the position of the server. This input will be

assumed Poisson with average interarrival time $1/\lambda$, so that the probability density of the time t between arrival of units on the queue is $\lambda e^{-\lambda t}$. The queues are served in cyclic order, i.e., they can be numbered $1, 2, \dots, N$ such that queue $(j+1)$ is served immediately after queue j and queue 1 immediately after queue N . At any queue, the server serves all units which were on that queue at the moment that he arrives. Units entering a queue while it is being served must wait until the server has made a complete cycle of the system before they can receive service. We denote by $s(t)$, the probability density of the time required to service a unit (with mean service time $1/\mu$) and we let $w(t)$ denote the probability density of the *walking time*. This is the time required by the server to move between queues; that is, the time between the completion of service at one queue and the inception of service at the next. In particular $w(t)$ (or $s(t)$) may be delta functions, i.e., the walking (or service) time may be constant. The functions λ , $s(t)$ and $w(t)$ will be assumed the same for each queue. Finally, no limit is placed on the length of the queues.

Actually, the treatment given here can be considerably extended. For example, at the expense of some additional complication, λ , $s(t)$ and $w(t)$ may be allowed to vary from queue to queue. We can also consider the case of queues served in random rather than cyclic order, or the case where a limit is imposed on the number of units that may be served at each queue. Some of these extensions will be taken up in a later paper.

In the application that brought this problem to our attention, the server was in fact a computer; the units, messages; the queues, the messages stored in a terminal;

the service time, the time required to transmit a message; and the walking time, the time needed to inform a terminal that it may begin to transmit its messages to the computer for processing.

The problem discussed here is similar to that of the "patrolling repairman"² except that in that case, only one unit is allowed on a queue.

We will consider here $\{p_n\}$ the stationary probability distribution of the number of units on a given queue at the moment of the server's arrival. Knowing this distribution, and the waiting time distribution, the efficiency of the system may be obtained. These will, however, be discussed elsewhere. In Section 2, as an orientation, we treat exactly the case $N=1$. In Section 3, we allow N to be arbitrary and we present an approximate method for finding $\{p_n\}$. This method apparently gives correctly the terms through $(\lambda/\mu)^3$ of a power series expansion of p_n in (λ/μ) with N fixed. Furthermore, we give, using this method, an asymptotic expansion of p_n in powers of $1/N$ with $N\lambda/\mu$ fixed. In Section 4, we compare our approximate method with an exact treatment of the case $N=2$.

2. Case of $N=1$

For the case³ $N=1$, let $P(n|n')$ denote the conditional probability that the server on his arrival finds n units on the queue, given that on his previous arrival he found n' . The quantities $P(n|n')$ ($n, n'=0, 1, \dots$) are the elements of a Markoff (stochastic) matrix. The stationary state probabilities p_n must satisfy the system of equations

$$p_n = \sum_{n'=0}^{\infty} P(n|n') p_{n'} \quad (1)$$

To obtain an expression for $P(n|n')$ consider the time that elapses between successive arrivals of the server at the queue. This time consists of two parts, (a) the time to serve n' units, and (b) the walking time.

The probability density of the time (a) is just the n' th fold convolution of the service time density, namely $s^{(n')} * (t)$; while the probability density of the time (b) is just $w(t)$. Hence, the probability density of the sum (a) + (b) is the convolution $s^{(n')} * (t) * w(t)$. During the time t , the probability that exactly n units enter the queue is $e^{-\lambda t} (\lambda t)^n / n!$

Thus, on integrating over all possible times we have

$$P(n|n') = \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!} s^{(n')} * (t) * w(t) dt \quad (2)$$

To solve the equations (1) we introduce the generating function

$$G(x) = \sum_{n=0}^{\infty} p_n x^n \quad (3)$$

Let $\Phi(r)$, $\Psi(r)$ denote the Laplace transforms of $s(t)$, $w(t)$ respectively:

$$\Phi(r) = \int_0^{\infty} e^{-rt} s(t) dt,$$

$$\Psi(r) = \int_0^{\infty} e^{-rt} w(t) dt \quad (4)$$

Multiply Eq. (1) by x^n , sum over n , and use Eq. (2); then

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} x^n P(n|n') p_{n'} \\ &= \sum_{n'=0}^{\infty} p_{n'} \int_0^{\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \\ &\quad \cdot x^n s^{(n')} * (t) * w(t) dt \\ &= \sum_{n'=0}^{\infty} p_{n'} \int_0^{\infty} e^{-\lambda t (1-x)} s^{(n')} * (t) * w(t) dt \\ &= \sum_{n'=0}^{\infty} p_{n'} \{ \Phi[\lambda(1-x)] \}^{n'} \Psi[\lambda(1-x)] \end{aligned} \quad (5)$$

Hence

$$G(x) = \Psi[\lambda(1-x)] G(z), \quad (6)$$

where

$$z = \Phi[\lambda(1-x)] \quad (7)$$

We have used the fact that the Laplace transform of the convolution of two functions is the product of their Laplace transforms.

The property of z that will prove crucial is that

$$z|_{x=1} = 1 \quad (8)$$

by Eqs. (4) and (7). This will enable us to calculate successively the moments of the distribution $\{p_n\}$ even though an explicit solution of the functional Eq. (6) appears impossible.

In fact, let

$$F(x) = \log G(x) \quad (9)$$

so that

$$F(x) = \log \Psi[\lambda(1-x)] + F(z) \quad (10)$$

Denote by $\bar{w}^j$ the j th moment of the walking time and let $\bar{s}^2$ be the mean square service time. Finally, recall that $1/\mu$ is the mean service time. Then if $\bar{n}$, $\bar{n}^2$ are the first and second moments, respectively, of the distribution $\{p_n\}$, it is easy to verify from Eq. (9) that

$$F'(1) = \bar{n} \quad (11)$$

$$F''(1) = \bar{n}^2 - \bar{n} - \bar{n}^2 \quad (12)$$

$$\bar{n}^2 = F''(1) + F'(1) + [F'(1)]^2 \quad (13)$$

To obtain $F'(1)$, $F''(1)$ differentiate Eq. (10) with respect to x and then put $x=1$. Making use of Eq. (4) to express the derivatives of

$$\Phi[\lambda(1-x)], \Psi[\lambda(1-x)]$$

in terms of $\bar{w}$, $\bar{w}^2$, $\bar{s}$, $\bar{s}^2$, we find that

$$F'(1) = \lambda/\mu F'(1) + \lambda \bar{w} \quad (14)$$

$$F''(1) = \lambda^2 \bar{s}^2 F'(1) + (\lambda/\mu)^2 F''(1) + \lambda^2 \delta^2(w), \quad (15)$$

where $\delta^2(w) = \bar{w}^2 - \bar{w}^2$. Hence

$$\bar{n} = \frac{\lambda \bar{w}}{1 - (\lambda/\mu)} \quad (16)$$

$$\bar{n}^2 = \frac{\lambda^2 \delta^2(w) + \lambda^2 \bar{s}^2 \bar{n}}{1 - (\lambda/\mu)^2} + \bar{n} + \bar{n}^2. \quad (17)$$

From Eq. (16) we see that if the mean arrival rate λ is equal to or greater than the mean service rate μ , the mean queue length becomes infinite. A very similar result holds for the case of $N > 1$ queues. Actually, when $\lambda > \mu$, it seems highly probable that no stationary state exists; in fact, this was proved for a very similar queueing problem by Foster.⁴

3. Case of arbitrary N — approximate treatment

With more than one queue, a complete description of the system at the moment of the server's arrival at any queue requires not only a knowledge of the number of units n_1 on that queue but also of the number of units $n_2, \dots, n_N$ on the other queues. For this reason, the quantities p_n no longer constitute the stationary probability distribution of a Markoff process. In fact, one must consider the joint probability distribution $\{p_{n_1}, \dots, p_{n_N}\}$ of the variables $n_1, \dots, n_N$. While it is possible to find a transition matrix analogous to the matrix $P(n|n')$ and hence a set of linear equations for $p_{n_1}, \dots, p_{n_N}$, the elements of this matrix⁵ rapidly increase in complication with growing N . Hence, though p_n may be obtained by summing $p_{n_1}, \dots, p_{n_N}$ over $n_2, \dots, n_N$, this is apparently not a feasible way of finding p_n if N is large.

To motivate the treatment given here, consider the following heuristic method used for finding the mean $\bar{n} = \sum n p_n$.

Let T denote the mean time required by the server to make a complete cycle of the system; i.e., the time between the beginning of successive service periods at say queue 1. Then T is composed of N waiting times (with mean total duration $N\bar{w}$) and the sum of the service periods at each of the N queues. To find the mean duration of these service periods, we assume that (A) at each queue $\bar{n}$ units meet the server. It follows that

$$T = N(\bar{n}/\mu) + N\bar{w}.$$

During this time on the average $\bar{n} = T\lambda$ units enter the queue 1. Hence

$$\bar{n}/\lambda = N(\bar{n}/\mu) + N\bar{w}$$

or

$$\bar{n} = \frac{N\lambda\bar{w}}{1 - N\lambda/\mu}. \quad (18)$$

Note that for $N=1$, Eq. (18) reduces to (16). Actually, it can be shown that (18) is correct for all N , i.e. the assumption A above leads to the right result for $\bar{n}$. It

seems reasonable to introduce an analogous but stronger assumption and in this way obtain, hopefully, an approximation to the probability distribution $\{p_n\}$. Therefore, we assume that (B) at each queue the same probability distribution of units $\{p_n\}$ meets the server independently of each other.

From this assumption, we can find what the distribution $\{p_n\}$ must be by using the following consistency argument: If the server meets the same probability distribution of units at each of the queues on one cycle of the system beginning with queue 1, then the same distribution must meet him when he returns to queue 1. $\{p_n\}$ is thus a "self-consistent" distribution.⁶ Note that $\bar{n}$ was calculated in this way.

Before discussing the range of applicability of this approximation, we determine the equation for the generating function

$$G(x) = \sum_{n=0}^{\infty} p_n x^n.$$

It is almost identical with Eq. (6). Indeed,

$$G(x) = \{\Psi[\lambda(1-x)]G(z)\}^N, \quad (19)$$

where Ψ and z are given by Eqs. (4) and (7). To show this, observe that if n is the number of units which enter queue 1 during a complete cycle of the system (beginning with queue 1) and Δn_j ($j=1, 2, \dots, N$) is the number which enter during the time between the beginning of service at queue j and at queue $j+1$, then

$$n = \Delta n_1 + \Delta n_2 + \dots + \Delta n_N. \quad (20)$$

Following the derivation of Eq. (6), the generating function of the probability distribution of the random variables Δn_j is seen to be

$$\Psi[\lambda(1-x)]G(z).$$

Equation (19) is then a consequence of assumption B and Eq. (20).

Introducing, as before, $F(x) = \log G(x)$, the moments of the probability distribution $\{p_n\}$ may be calculated in the same way as in the previous section. We have

$$F(x) = N \log \Psi[\lambda(1-x)] + NF(z) \quad (21)$$

and

$$F'(1) = \frac{N\lambda}{\mu} F'(1) + \lambda\bar{w} \quad (22)$$

$$F''(1) = N\lambda^2 \bar{s}^2 F'(1) + N \left(\frac{\lambda}{\mu} \right)^2 F''(1) + N\lambda^2 \delta^2(w). \quad (23)$$

Hence

$$\bar{n} = \frac{N\lambda\bar{w}}{1 - N(\lambda/\mu)} \quad (24)$$

$$\bar{n}^2 = \frac{N\lambda^2 \bar{s}^2 \bar{n} + N\lambda^2 \delta^2(w)}{1 - N(\lambda/\mu)^2} + \bar{n} + \bar{n}^2. \quad (25)$$

Note that Eq. (24) agrees with Eq. (18). For a finite mean (and most likely, for a stationary state as well) we must have

$$N\lambda/\mu < 1. \quad (26)$$

This means that the time interval between the arrival of units at a given queue must be less than the time to serve one unit for each queue.

Consider now the case of fixed N and small λ/μ . For convenience, we choose our time scale such that $\mu=1$ so that the denominators in (24) and (25) become just $1-N\lambda$, $1-N\lambda^2$, respectively. However, we retain the same notation for $\bar{s}^2$ and the moments of the walking time distribution. This choice of time scale does not, of course, affect the values of $\bar{n}$ and $\bar{n}^2$ since they are nondimensional.

It will be shown that we can obtain an expansion for p_n in powers of λ .

First, note that $F'(1)$ is of order λ and $F''(1)$ is of order λ^2 . This follows from (22) and (23) with $\mu=1$. It is easy to see from Eq. (21) that each successive derivative of $F(x)$ at $x=1$ involves a higher power of λ , so that, for example

$$F^{(j)}(x)|_{x=1} \text{ is } O(\lambda^j).$$

Furthermore, note that the term in λ^3 in $F'''(1)$ comes entirely from $N \log \Psi[\lambda(1-x)]$. Hence, to terms of $O(\lambda^3)$ we have

$$F'''(1) = \lambda^3 (\bar{w}^3 - 3\bar{w}\bar{w}^2 + 2\bar{w}^3). \quad (27)$$

Since

$$F(x) = -F'(1)(1-x) + \frac{F''(1)}{2}(1-x)^2 - \frac{F'''(1)}{6}(1-x)^3 + \dots \quad (28)$$

it follows that to terms of $O(\lambda^3)$

$$F(x) = \frac{-N\lambda w}{1-N\lambda}(1-x) + \left[\frac{N^2\lambda^3 w \bar{s}^2}{1-N\lambda} + N\lambda^2 \delta^2(w) \right] \frac{(1-x)^2}{2} - N\lambda^3 (\bar{w}^3 - 3\bar{w}\bar{w}^2 + 2\bar{w}^3) \frac{(1-x)^3}{6}. \quad (29)$$

The highest order term in $F(x)$ is due entirely to the randomness of the input to queue 1, i.e., $F(x)$ would be exactly

$$\frac{-N\lambda w}{1-N\lambda}(1-x)$$

if the time required by the server to cycle the system were fixed at $T=\bar{n}/\lambda$.

The term $N\lambda^2 \delta^2(w) (1-x)^2/2$ has its origin in the variance of the walking time. It is interesting to observe that the effect of a nonconstant service time enters for

the first time in the λ^3 term.

Since $G(x) = \exp[F(x)]$, we can obtain p_n by using Eq. (29) and expanding in a power series in x . We find, to terms in λ^3 ,

$$p_0 = 1 + a\lambda + [(a^2/2) + b]\lambda^2 + [c + d + (a^3/6) + ab]\lambda^3 \quad (30)$$

$$p_1 = -\{a\lambda + (a^2 + 2b)\lambda^2 + [2c + 3d + (a^3/2) + 3ab]\lambda^3\}$$

$$p_2 = [(a^2/2) + b]\lambda^2 + [c + 3d + (a^3/2) + 3ab]\lambda^3$$

$$p_3 = -[d + (a^3/6) + ab]\lambda^3$$

$$p_n = 0 \quad \text{for } n > 3,$$

where

$$a = \frac{-N\bar{w}}{1-N\lambda}, \quad b = \frac{N}{2} \delta^2(w), \quad c = \frac{N^2}{2} \frac{\bar{w}\bar{s}^2}{1-N\lambda}$$

$$d = -\frac{N}{6} (\bar{w}^3 - 3\bar{w}\bar{w}^2 + 2\bar{w}^3).$$

It will be shown in the next section that for $N=2$ Eq. (30) does, in fact, give the probabilities p_n correct to terms in λ^3 . It is reasonable to expect that it should be valid for all N . It is natural to suppose that our method should give good results for small λ , since then the input to the system is small, the walking time is the predominant factor, and the dependence of the queues on each other is relatively weak.

There is another case for which one would expect our treatment to be applicable, and that is when N is large. Then, though assumption B is violated at a number of queues, these violations should cancel one another so that assumption B holds in some average sense. But this is sufficient for our method to be valid since the influence of any one queue on another is small for large N .

Consider, then, the case of large N with $\alpha \equiv N\lambda$ fixed. We can verify, as above, that correct to terms in $1/N$

$$F(x) = \frac{\alpha \bar{w}}{1-\alpha}(1-x) + \frac{1}{N} \left[\frac{\alpha^3 \bar{s}^2 \bar{w}}{1-\alpha} + \alpha^2 \delta^2(w) \right] \frac{(1-x)^2}{2} + \dots \quad (31)$$

Hence expanding $\exp[F(x)]$ in a power series in x we find to terms $O(1/N)$

$$p_0 = e^{-f} \left(1 + \frac{g}{2N} \right)$$

$$p_1 = e^{-f} \left(f + \frac{gf}{2N} - \frac{g}{N} \right)$$

$$p_n = e^{-f} \frac{f^n}{n!} + \frac{g}{N} e^{-f} \frac{f^{n-2}}{(n-2)!} \left(\frac{f^2}{2} - nf + \frac{n(n-1)}{2} \right), \quad \text{for } n \geq 2 \quad (32)$$

where

$$f = \frac{\alpha \bar{w}}{1-\alpha}, g = \frac{\alpha^3 s^2 \bar{w}}{1-\alpha} + \alpha^2 \delta^2(w).$$

Note that in this case $p_n > 0$ for all n , and that in the limit of infinite N , the distribution $\{p_n\}$ reduces to a Poisson distribution with the parameter f .

4. Exact treatment of the case $N=2$

In order to compare with the results of the previous section, we treat *exactly* the case $N=2$. Let p_{nm} denote the joint probability that in a stationary state there are n and m units in queues 1 and 2, respectively at the moment when the server arrives at queue 1; and let $P(n, m, |n', m')$ be the corresponding probabilities given that there were n', m' units in queues 1 and 2 at the server's last previous arrival at queue 1. Then

$$p_{nm} = \sum_{n', m'=0}^{\infty} P(n, m | n', m') p_{n'm'} \quad (33)$$

To find the "transition" probabilities $P(n, m | n', m')$ we proceed as follows:

Let $D(t_1, t_2, l | n', m')$ be the conditional joint probability density that, given there are n', m' units in queues 1 and 2, respectively, when the server begins service at queue 1,

- (a) a time t_1 is required to service queue 1 and move on to queue 2
- (b) there are l units in queue 2 when the server arrives
- (c) it takes a time t_2 to service queue 2 and return to queue 1.

In a manner similar to that used in obtaining $P(n | n')$ in Sec. 2, we have

$$D(t_1, t_2, l | n', m') = s^{(n')} * (t_1) * w(t_1) e^{-\lambda t_1} \frac{(\lambda t_1)^{l-m'}}{(l-m')!} s^{(l)} * (t_2) * w(t_2) \quad (34)$$

From the definition of $D(t_1, t_2, l | n', m')$ it follows that

$$P(n, m | n', m') = \sum_{l=m'}^{\infty} \int_0^{\infty} \int_0^{\infty} D(t_1, t_2, l | n', m') \cdot e^{-\lambda(t_1+t_2)} [\lambda(t_1+t_2)]^n e^{-\lambda t_2} \frac{(\lambda t_2)^m}{m!} dt_1 dt_2 \quad (35)$$

To solve Eq. (33) with $P(n, m | n', m')$ given by Eq. (35) we again use the method of generating functions.

Let

$$G(x, y) = \sum_{n, m=0}^{\infty} p_{nm} x^n y^m \quad (36)$$

Then multiplying Eq. (33) by $x^n y^m$ and summing over n and m , we obtain after some calculation

$$G(x, y) = \Psi(u) \Psi(v) G[\Phi(u), \Phi(v)], \quad (37)$$

where

$$u = \lambda(2-x-y) \quad (38)$$

$$v = \lambda(1-x) + \lambda - \lambda \Phi(u). \quad (39)$$

In verifying Eq. (37), one sums the terms

$$\frac{[\lambda(t_1+t_2)]^n}{n!} x^n \frac{(\lambda t_2)^m}{m!} y^m$$

over n, m respectively, in Eq. (33), then integrates over t_2 , sums over l , integrates over t_1 , and finally sums over m' and n' . The details are analogous to those met in deriving Eq. (6).

Again, while an explicit solution of Eq. (37) seems far too much to hope for, the moments of the probability distribution can be found by using the fact that

$$\Phi(u) |_{x=y=1} = 1, \Phi(v) |_{x=y=1} = 1.$$

Let $F(x, y) = \log G(x, y)$ so that

$$F(x, y) = \log \Psi(u) + \log \Psi(v) + F(\Phi(u), \Phi(v)). \quad (40)$$

If we define

$$\bar{n} = \sum_{n, m=0}^{\infty} n p_{nm}, \bar{n}^2 = \sum_{n, m=0}^{\infty} n^2 p_{nm}$$

then $\bar{n}$ and $\bar{n}^2$ are just the first and second moments of the probability distribution $\{p_n\}$ considered in the previous section, since

$$p_n = \sum_{m=0}^{\infty} p_{nm}.$$

To find $\bar{n}$ and $\bar{n}^2$, we have

$$\bar{n} = F_x(1, 1) \quad (41)$$

$$\bar{n}^2 = F_{xx}(1, 1) + \bar{n} + \bar{n}^2. \quad (42)$$

Differentiating Eq. (40) with respect to x and y and then putting $x=y=1$, we obtain a set of linear equations for the first and second derivatives of $F(x, y)$ at $x=y=1$. After some extensive but fairly straightforward computation, we find with time units adjusted so that $\mu=1$:

$$F_x = \frac{2\bar{w}\lambda}{1-2\lambda} \quad (43)$$

$$DF_{xx} = [\delta^2(w) + (1/2)\bar{n}\bar{s}^2]\lambda^2(1+\lambda^2) + (\bar{s}^2\bar{w} + \bar{n}\bar{s}^2)\lambda^3(1+\lambda^2) + [\delta^2(w) + \bar{n}\bar{s}^2]\lambda^2(-4\lambda^3 - \lambda^2 + 2\lambda + 1), \quad (44)$$

where $D = (1-2\lambda)(1-\lambda^2)(2\lambda^2+2\lambda+1)$.

For small λ , we have correct to terms of $O(\lambda^3)$

$$F_{xx} = 4\lambda^3 \bar{w} \bar{s}^2 + 2\lambda^2 \delta^2(w), \quad (45)$$

using Eq. (43) for $\bar{n}$.

To $O(\lambda^3)$, Eq. (45) is just the same as $F_x(1)$ in the last section with $N=2$.

Furthermore, note that the logarithm of the generating function of the probability distribution $\{p_n\}$ is just $F(x, 1)$. If we expand $F(x, 1)$ around $x=1$,

$$F(x, 1) = -F_x(1, 1)(1-x) + \frac{F_{xx}(1, 1)}{2}(1-x)^2 - \frac{F_{xxx}(1, 1)}{6}(1-x)^3 + \dots \quad (46)$$

the only coefficient which is not $O(\lambda^3)$, [besides $F_x(1, 1)$ and $F_{xx}(1, 1)$] is $F_{xxx}(1, 1)$; and as can be seen by further differentiating Eq. (40) the term in λ^3 in $F_{xxx}(1, 1)$ is just

$$\lambda^3(\bar{w}^3 - 3\bar{w}\bar{w}^2 + 2\bar{w}^3). \quad (47)$$

On comparing now Eq. (46) with Eq. (29) (the coefficients in (46) being given by Eqs. (43), (45), (47)), we see that they agree to $O(\lambda^3)$. Hence it follows that the probability distribution $\{p_n\}$ as calculated from the exact generating function $\exp[F(x, 1)]$ agree to $O(\lambda^3)$ with the distribution obtained using our approximate method.

5. Conclusions

An approximate method based on the idea of a "self-consistent" probability distribution has been used to analyze a multiqueue system. This method is at least reminiscent of self-consistent field approximations in atomic physics.

In general, the growing complication of the queueing structures found in modern communication systems

means that exact analyses will be increasingly difficult to obtain; suitable approximate techniques must therefore be developed. As a source of clues to what these techniques might be, we suggest the extensive theory of many-body systems in mathematical physics.

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References and footnotes

1. For an introduction, see P. M. Morse, *Queues, Inventories and Maintenance*, John Wiley, 1958.
2. A. T. Barucha-Reid, *Elements of the Theory of Markov Processes and Their Applications*, McGraw-Hill, 1960. Page 419.
3. To give an intuitive meaning to the walking time in the case $N=1$, we can imagine the server collecting the units, transporting them to some remote point for service, and then returning to the queue. The time taken traveling to and from this remote point may be regarded as the walking time.
4. F. G. Foster, *Ann. Math. Stat.* **24**, 355 (1953).
5. They will be given explicitly for $N=2$ in the next section.
6. Compare with the self-consistent approximations used in atomic physics, viz., L. Landau and E. Lifschitz, *Quantum Mechanics*, Addison-Wesley, 1957.

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